We learn **deep generative models** whose representations are **invariant** under symmetry transformations.

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Learning Deep Generative Models with Invariance under Symmetry Transformations

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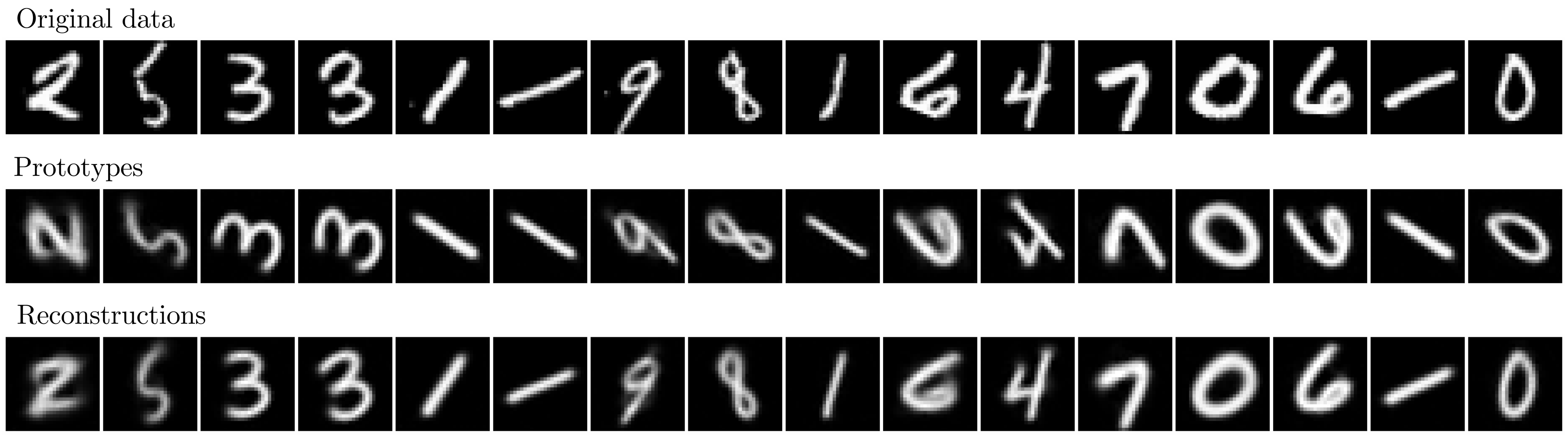
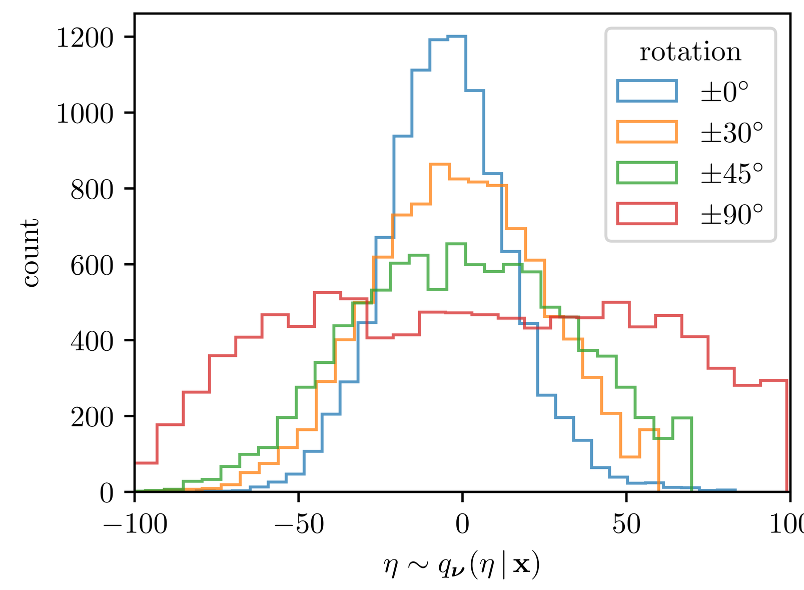
We learn representations of MNIST digits that are invariant to rotations and successfully reconstruct the original digits:



As digits are rotated more, we learn to predict larger angles:

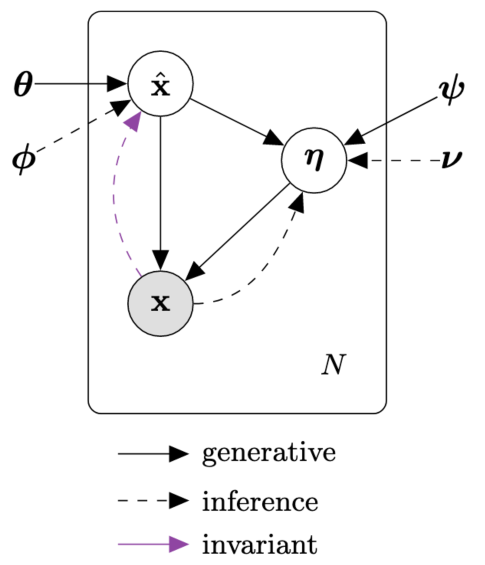


Our model:

$$\hat{\mathbf{x}} \sim p_{\boldsymbol{\theta}}(\hat{\mathbf{x}}), \quad (1)$$

$$\boldsymbol{\eta} \sim p_{\boldsymbol{\psi}}(\boldsymbol{\eta} \,|\, \hat{\mathbf{x}}), \quad (2)$$

$$\mathbf{x} \sim p(\mathbf{x} \,|\, \boldsymbol{\eta}, \hat{\mathbf{x}}), \quad (3)$$



Affine transformation parameterization:

$$\mathcal{T}_{\boldsymbol{\eta}}(\hat{\mathbf{x}}) = T_{\boldsymbol{\eta}} \cdot \hat{\mathbf{x}}, \quad T_{\boldsymbol{\eta}} = \exp\left(\sum_i \eta_i G_i\right) \quad (4)$$

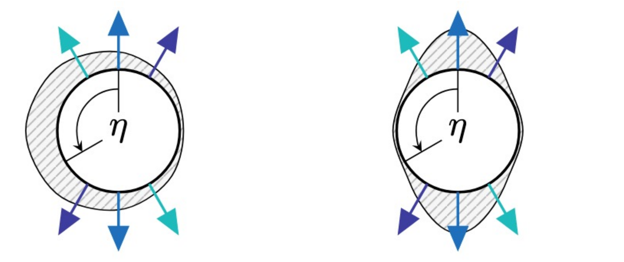
Our training objective:

$$\log p(\mathbf{x}) = \log \iint p(\mathbf{x}, \hat{\mathbf{x}}, \boldsymbol{\eta})\, d\boldsymbol{\eta}\, d\hat{\mathbf{x}} \quad (5)$$

$$= \log \mathop{\mathbb{E}}_{q_{\nu}(\boldsymbol{\eta} \,|\, \mathbf{x})\, q_{\phi}(\hat{\mathbf{x}} \,|\, \mathbf{x})} \left[\frac{p(\mathbf{x} \,|\, \hat{\mathbf{x}}, \boldsymbol{\eta})\; p_{\psi}(\boldsymbol{\eta} \,|\, \hat{\mathbf{x}})\; p_{\boldsymbol{\theta}}(\hat{\mathbf{x}})}{q_{\boldsymbol{\nu}}(\boldsymbol{\eta} \,|\, \mathbf{x})\; q_{\boldsymbol{\phi}}(\hat{\mathbf{x}} \,|\, \mathbf{x})}\right] \quad (6)$$

$$\geq \mathop{\mathbb{E}}_{q_{\boldsymbol{\nu}}\, q_{\boldsymbol{\phi}}} [\log p(\mathbf{x} \,|\, \hat{\mathbf{x}}, \boldsymbol{\eta})] - \mathop{\mathbb{E}}_{q_{\boldsymbol{\phi}}} [D_{\mathrm{KL}}(q_{\boldsymbol{\nu}} \,||\, p_{\psi})] - D_{\mathrm{KL}}(q_{\boldsymbol{\phi}} \,||\, p_{\boldsymbol{\theta}}) \equiv -\mathcal{L}(\boldsymbol{\theta}, \boldsymbol{\psi}, \boldsymbol{\phi}, \boldsymbol{\nu}) \quad (7)$$

Conjecture 1: $p_{\boldsymbol{\psi}}(\eta \,|\, \hat{\mathbf{x}})$ must be sufficiently flexible.

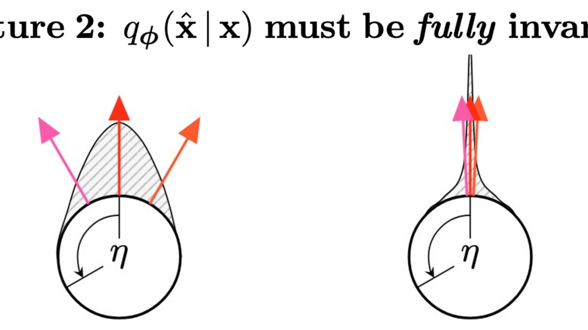


(a) Simple $p_{\boldsymbol{\psi}}(\eta \,|\, \hat{\mathbf{x}})$ (b) Flexible $p_{\boldsymbol{\psi}}(\eta \,|\, \hat{\mathbf{x}})$

$p_{\boldsymbol{\psi}}(\eta \,|\, \hat{\mathbf{x}})$—▨—and $p(\eta \,|\, \hat{\mathbf{x}})$—↑↑↑.

$\mathbf{x}$	$\hat{\mathbf{x}}$	$p(\eta \,\|\, \mathbf{x}, \hat{\mathbf{x}})$
8	8	$0.5 \cdot \delta(\eta - 0°) + 0.5 \cdot \delta(\eta - 180°)$
8	8	$0.5 \cdot \delta(\eta - 30°) + 0.5 \cdot \delta(\eta + 150°)$
8	8	$0.5 \cdot \delta(\eta + 30°) + 0.5 \cdot \delta(\eta - 150°)$

Conjecture 2: $q_{\boldsymbol{\phi}}(\hat{\mathbf{x}} \,|\, \mathbf{x})$ must be *fully* invariant w.r.t η.

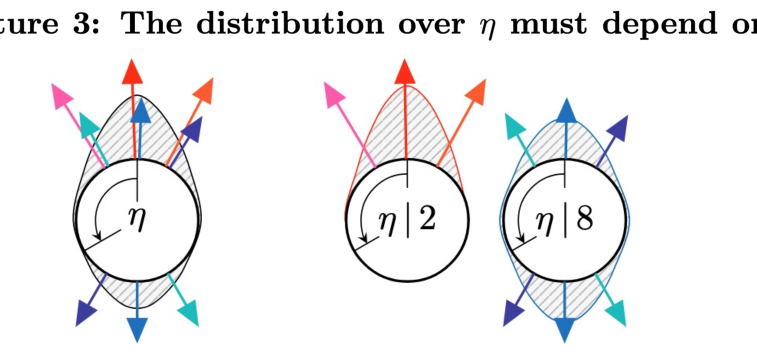


(a) $\rho \geq 60°$ (b) $60° > \rho$

$p_{\boldsymbol{\psi}}(\eta \,|\, \hat{\mathbf{x}})$—▨—and $p(\eta \,|\, \hat{\mathbf{x}})$—↑↑↑.

	$\rho \geq 60°$		$60° > \rho$	
$\mathbf{x}$	$\hat{\mathbf{x}}$	$p(\eta \,\|\, \mathbf{x}, \hat{\mathbf{x}})$	$\hat{\mathbf{x}}$	$p(\eta \,\|\, \mathbf{x}, \hat{\mathbf{x}})$
2	2	$\delta(\eta - 0°)$	2	$\delta(\eta - 0°)$
2	2	$\delta(\eta - 30°)$	2	$\delta(\eta - 0°)$
2	2	$\delta(\eta + 30°)$	2	$\delta(\eta - 0°)$

Conjecture 3: The distribution over η must depend on $\hat{\mathbf{x}}$.



(a) η (b) $\eta \,|\, \hat{\mathbf{x}}$

$p_{\boldsymbol{\psi}}(\eta)$ or $p_{\boldsymbol{\psi}}(\eta \,|\, \hat{\mathbf{x}})$—▨—and $p(\eta)$ or $p(\eta \,|\, \hat{\mathbf{x}})$—↑↑↑↑↑↑.

$\mathbf{x}$	$\hat{\mathbf{x}}$	$p(\eta \,\|\, \mathbf{x}, \hat{\mathbf{x}})$
2	2	$\delta(\eta - 0°)$
2	2	$\delta(\eta - 30°)$
2	2	$\delta(\eta + 30°)$
8	8	$0.5 \cdot \delta(\eta - 0°) + 0.5 \cdot \delta(\eta - 180°)$
8	8	$0.5 \cdot \delta(\eta - 30°) + 0.5 \cdot \delta(\eta + 150°)$
8	8	$0.5 \cdot \delta(\eta + 30°) + 0.5 \cdot \delta(\eta - 150°)$